Portfolio Optimization under Value-at-Risk Constraints

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(Joint work with **Prof Tapen Sinha, ITAM, Mexico**)

1 Introduction

Portfolio diversification has been a theme for the ages.

In the Merchant of Venice, William Shakespeare had Antonio say: *My ventures are not in one bottom trusted, Nor to one place; nor is my whole estate Upon the fortune of this present year*

Introduction ...

A similar sentiment was echoed by R. L. Stevenson in Treasure Island (1883), where Long John Silver commented on where he keeps his wealth, *I puts it all away, some here, some there, and none too much anywheres?*

3 Introduction ...

Not all writers had the same belief about diversification.

For example, Mark Twain had Pudd?nhead Wilson say: *Put all your eggs in the one basket and ? watch that basket* (Twain, M., 1893, chap. 15). Curiously, Twain was writing the novel to sell it to stave off bankruptcy.

Introduction ...

Prior to Markowitz's work, some investors focused on assessing the risks and rewards of individual securities in constructing their portfolios.

One view was to identify those securities that offered the best opportunities for gain with the least risk and then construct a portfolio from these.

5 Markowitz

Markowitz proposed that mean return be taken as a proxy for *reward* while the standard deviation be taken as a proxy for *risk*.

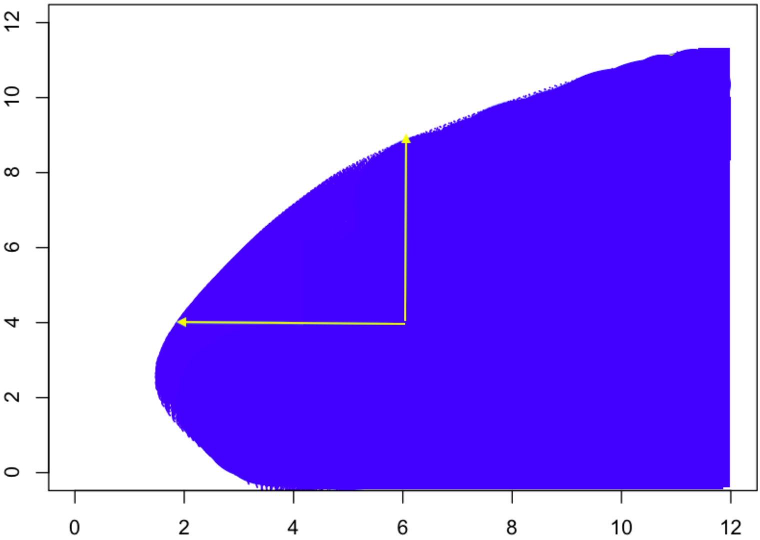
He argued that for a given level of return, an investor should look for a portfolio that minimizes the standard deviation or if she is comfortable with a given level of risk as measured by standard deviation, then she should look for a portfolio that maximizes the return (mean).

6 Markowitz...

The next figure shows the risk-return plot for a data on 5 stocks.

Here the mean and standard deviation are expressed in basis points (bp) - 1/10000 or 0.01

The points represent risk and return for a chosen portfolio.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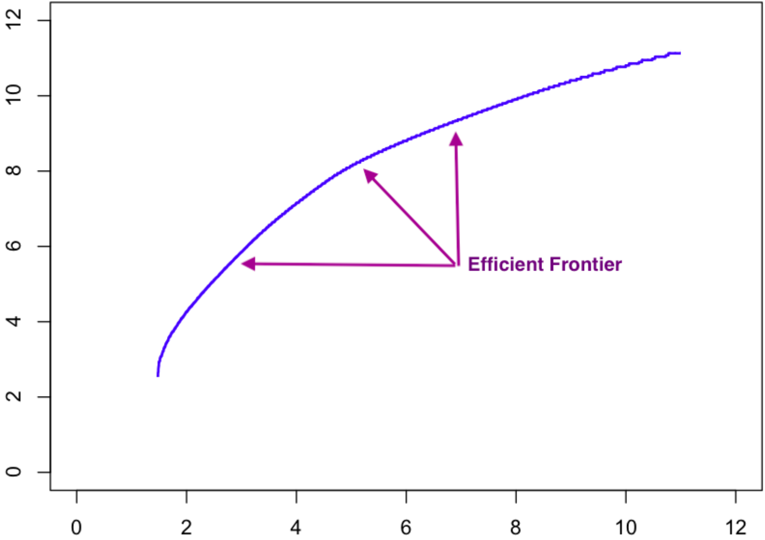
Markowitz...

Thus every investor should choose a portfolio from among the ones appearing in the next figure, called the efficient frontier.


Efficient Frontier
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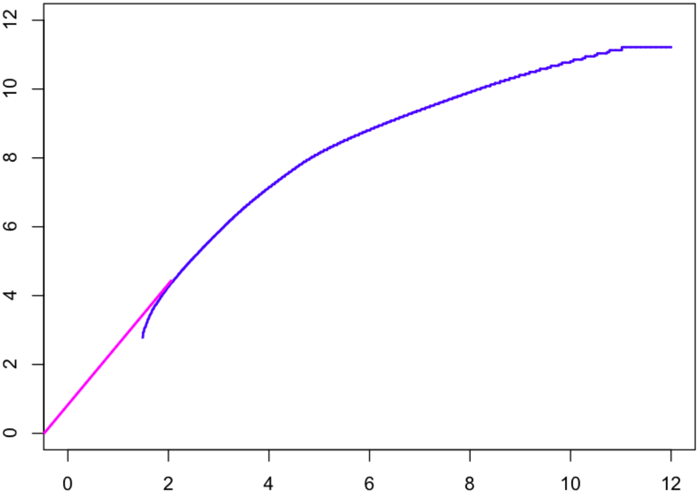
Markowitz...

Tobin argued that the Efficient frontier could be improved upon by adding cash (zero return, zero risk) to the portfolio.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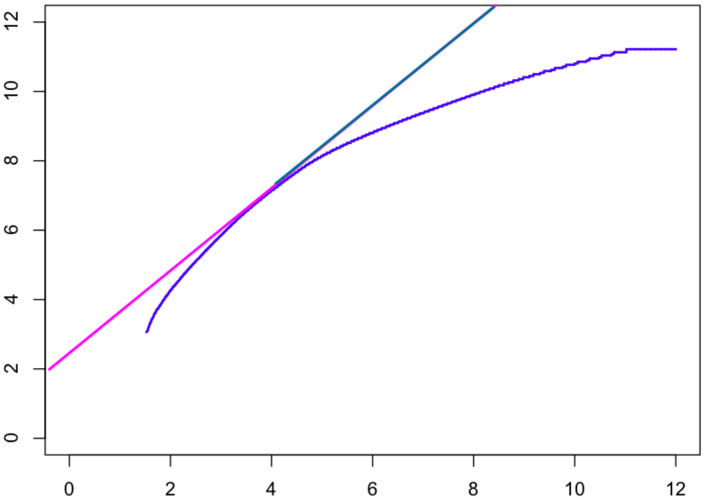
12 Markowitz...

Sharp proposed adding risk free asset (zero risk bonds/ treasury bills) to produce portfolios that improve upon the Markowitz efficient frontier:



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Markowitz...

Now the efficient frontier is represented by the line that is a tangent to the Markowitz efficient frontier which intersects the vertical axis at the level of the risk-free asset.
This line is called *Capital Market Line*. The slope of the Capital Market line is the *Sharp Ratio*:

$$\sigma_s = \max_P \frac{m(P) - r_f}{\sigma(P)}$$

where $m(P), \sigma(P)$ are the mean and standard deviation of returns on a portfolio P and the maximum is taken over all portfolios.

Markowitz...

This analysis intrinsically assumes that the returns on stocks follow multivariate normal distribution. Then the returns on any portfolio would follow normal distribution and then the risk can be measured by its standard deviation.

16 Beyond Markowitz and Normal distribution

For decades it has been noted that the returns on stocks mostly do not follow normal distribution. In most cases, they tend to have fatter tails than normal distribution. In this case, standard deviation may not be a good measure of risk.
As a result, for risk management purposes, Value-at-Risk (VaR) has been accepted as a measure of risk and is now part of international regulations on risk management.

Beyond Markowitz and Normal distribution

However, standard deviation as a measure of risk is continued to be used when it comes to optimal portfolio selection via Markowitz paradigm.

When we move away from Gaussian or normal distribution, we could also replace mean by median as a measure for return and the gap between median and VaR (say 5%) as a measure for risk.

Beyond Markowitz and Normal distribution ...

Recall that 5% VaR for a portfolio P would be the 95^{th} percentile of the distribution of $-R(P)$, where $R(P)$ is the return on the portfolio P, so that $-R(P)$ is the loss.

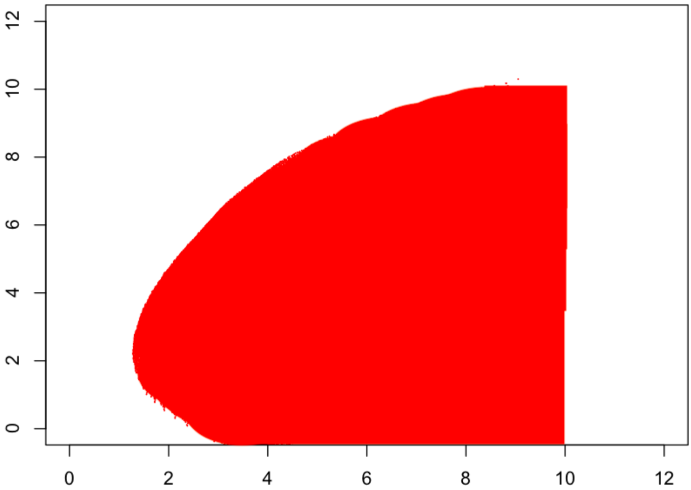
Thus we use gap between 50^{th} percentile and 5^{th} percentile of distribution of $R(P)$, in other words, *(median + VaR)* as the measure of risk.

Beyond Markowitz and Normal distribution ...

Instead of plotting *(median + VaR)* vs *median*, in order to be compatible with Markowitz paradigm, we proceed as follows. For a normal distribution, the 5% VaR is 1.644854σ away from the mean or median. Thus we plot

(median + VaR)/1.644854 on x-axis

and $median$ on y-axis for the portfolios considered earlier to get the following picture: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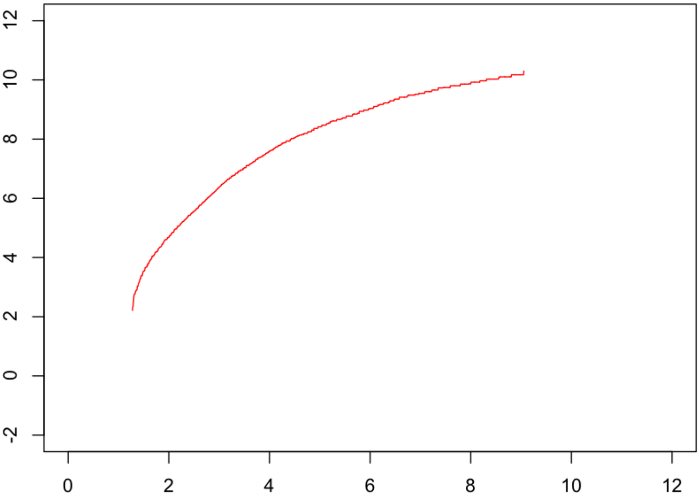
Beyond Markowitz and Normal distribution ...

Now following Markowitz, we could argue that the analogue of the efficient frontier in this paradigm is given by



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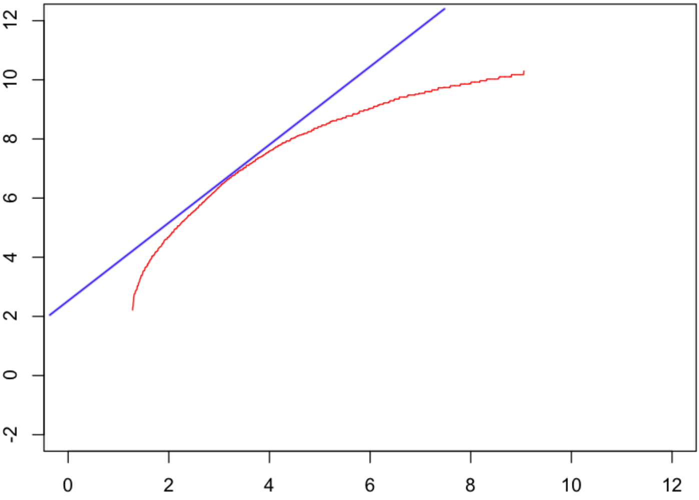
Beyond Markowitz and Normal distribution ..

and the analogue of the Capital Market Line in this paradigm is given by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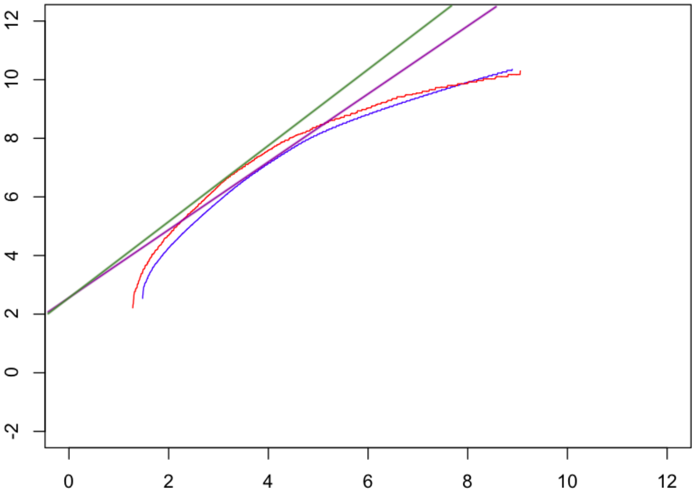
Beyond Markowitz and Normal distribution ..

Here are the two graphs in a single figure.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Beyond Markowitz and Normal distribution ..

The analogue of the Sharp Ratio here is the τ_R - Tau-ratio, defined as follows:

For a portfolio P, let $\alpha(P)$ denote the median of the distribution of the return for the portfolio P and let $\beta_5(P)$ denote its 5^{th} percentile. The 5% VaR is then $-\beta_5(P))$. The measure of risk is

$$\theta(P) = \frac{1}{1.644854}(\alpha(P) - \beta_5(P))$$

Beyond Markowitz and Normal distribution ..

and the risk-return measure for the portfolio P is

$$\Gamma(P) = \frac{\alpha(P) - r_f}{\theta(P)}$$

The Tau-ratio is:

$$\tau_R = \sup_P \Gamma(P).$$

If the returns follow multivariate normal distribution, $\tau_R = \sigma_S$.

Computational issues ..

If the number of stocks is in hundreds or in thousands, computation of τ_R based on historical data and the corresponding portfolio is a difficult task.
The sample quantiles of a portfolio are not convex functions of the weights and thus it is a computationally a difficult problem.

Computational issues ..

We have come up with heuristics that seem to improve significantly $\Gamma(P)$ as compared to the Γ of optimal portfolio achieving the Sharp Ratio.

We start with the optimal portfolio achieving the Sharp Ratio for the given riskfree rate r_f as well as for rates varying between $\frac{r_f}{2}$ and $2r_f$. These are our initial set of portfolios.

Computational issues ..

Also, we tweak the historical data to elongate the left tails of the stocks and get the optimal portfolios for the modified historical data, getting alternate set of portfolios.

Computational issues ..

Starting with the initial portfolios, we consider convex combinations of each of these with alternate portfolios and, if it improves Γ, update the initial portfolio and proceed.

Also we consider convex combinations of the (updated) initial portfolios with individual stocks and update, if Γ improves.

We have done experiments with real market data: data on 496 stocks (out of 500 in SP) - for over 2 years - 2016 and 2017.

Computational issues ..

For the riskfree rate $r_f = 0.0001$, the optimal portfolio that achieves the sharp ratio has $\Gamma(P) = 0.146537$.

The heuristic algorithm that we have developed yields $\Gamma(P) = 0.187125$, a gain of about 30%. It may not be optimal as we have no means of checking this. On the other it provides significant improvement on the optimal portfolio in the Markowitz-Sharp framework as far as Γ is concerned.

Computational issues ..

The current algorithm is taking about 1 hour on a stand alone machine.

We have tried running the same with finer grid for convex combinations and have run it for 12 hours, but have not seen improvement as yet as far as maximum $\Gamma(P)$ is concerned.

But there is scope for improvement of the algorithm and we are working on the same.